

CLASSIFICATION OF FREE ACTIONS ON COMPLETE INTERSECTIONS OF FOUR QUADRICS

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ABSTRACT. In this paper we classify all free actions of finite groups on Calabi-Yau complete intersection of 4 quadrics in \mathbb{P}^7 , up to projective equivalence. We get some examples of smooth Calabi-Yau threefolds with large nonabelian fundamental groups. We also observe the relation between some of these examples and moduli of polarized abelian surfaces.

1. INTRODUCTION

The original motivation of this paper is to generalize Beauville's construction of Calabi-Yau manifolds with a non-abelian fundamental group[B]. As one result of this paper, we construct many new examples of Calabi-Yau manifolds with non-abelian fundamental groups. In particular we construct five families of Calabi-Yau threefolds with fundamental groups of order 64. All these families are related to pencils of certain abelian surfaces. Three of these families have been previously studied in [GPa] and [BH]. The new examples are constructed as free quotients of small resolutions of singular complete intersections of four quadrics in \mathbb{P}^7 that contain a pencil of (2,4) polarized abelian surfaces (theorem 7.8).

We also classify all families of complete intersections of four quadrics in \mathbb{P}^7 with a free finite group action and at most ODP singularities. The key idea is to use Holomorphic Lefschetz formula to obtain restriction on possible group actions. This paper is quite elementary, the reasoning is sometimes very explicit and is never very deep. Calculations of this paper can be generalized to other complete intersections in projective spaces or in products of projective spaces.

The paper is organized as follows. In section 2, we review the construction of a smooth Calabi-Yau 3-fold with quaternion group H_8 acting freely on it due to Beauville. We will see how the character theory of H_8 and Holomorphic Lefschetz formula make this the only possible family of complete intersections with H_8 action. We also see that no linear action of the dihedral group D_8 could lead to any similar examples. In section 3, we give a brief introduction to projective

representation of finite groups and define the terminology of allowable actions, semi-allowable actions and Lefschetz condition. Section 4 contains a scheme of the algorithm of classifying (semi-)allowable actions on complete intersections of four quadrics in \mathbb{P}^7 . As an application we make several tables in the next section, listing all the (semi-)allowable actions with groups of order from 2 to 64. In section 6 we compute the cut out equations of families of Calabi-Yau 3-folds with order 64 semi-allowable actions. There are two such families with five different order 64 semi-allowable actions. In the last section we prove the existence of equivariant small resolutions (6.1 and 6.3). We also explain the relations between these Calabi-Yau 3-folds and moduli of some polarized abelian surfaces.

All the group theoretic calculations are done in GAP[GAP]. The software package Macaulay 2[M] is also very useful to us in checking smoothness. I am grateful to my advisor Lev Borisov, who gave many important ideas for this project.

2. BEAUVILLE'S EXAMPLE

In this section we will first review Beauville's example of a free action of quaternion group H_8 on a nine dimensional family of smooth complete intersections of four quadrics in \mathbb{P}^7 [B]. Additionally we will explain why there is no such family with free action of the dihedral group D_8 . In the process we will see how holomorphic Lefschetz formula leads to restriction on possible free group actions.

The quaternion group H_8 is the group of order 8 with elements $\pm 1, \pm i, \pm j, \pm k$ and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. By character calculation, H_8 has 4 one dimensional irreps and 1 two dimensional irrep. We denote them by V_1, V_2, V_3, V_4 and W . The regular representation V has decomposition $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus W^{\oplus 2}$. The induced representation on the second symmetric product of V has decomposition $Sym^2(V) = V_1^{\oplus 5} \oplus V_2^{\oplus 5} \oplus V_3^{\oplus 5} \oplus V_4^{\oplus 5} \oplus W^{\oplus 8}$. Pick 4 generic quadrics q_1, \dots, q_4 where q_i belongs to V_i . For generic choice of q_i Beauville showed that the complete intersection X in $\mathbb{P}(V^*)$, given by $q_1 = \dots = q_4 = 0$ is smooth and action of H_8 on X has no fixed points. As a consequence the quotient variety X/H_8 is a smooth Calabi-Yau manifold with fundamental group H_8 .

The following theorem is a special case of the standard Holomorphic Lefschetz formula:

Theorem 2.1. *Let X be a smooth algebraic variety over \mathbb{C} and $f : X \rightarrow X$ be a holomorphic automorphism of finite order with no fixed*

points. For a linearized coherent sheaf \mathcal{F} , the Lefschetz number

$$\Lambda(f, \mathcal{F}) = \sum_{q=0}^m (-1)^q \text{Tr}(f^*; H^q(X, \mathcal{F}))$$

is zero, where Tr stands for the trace.

Holomorphic Lefschetz formula explains why Beauville needed to pick this particular representation V and these particular choices of quadrics q_i . We identify the vector space V with $H^0(X, \mathcal{O}(1))$. By Kodaira's vanishing theorem and Holomorphic Lefschetz formula, $\text{Tr}(g, H^0(X, \mathcal{O}(1)))=0$ for any g non-identity. Group H_8 has 5 conjugacy classes represented by $\{(1), (i), (j), (-1), (k)\}$. Computing traces of each conjugacy class, we get the trace vector $[8, 0, 0, 0, 0]$ for V , which means it must be the regular representation. The induced representation $\text{Sym}^2(V)$ has trace vector $[36, 0, 0, 4, 0]$. By Lefschetz formula $H^0(X, \mathcal{O}(2))$ has trace vector $[32, 0, 0, 0, 0]$. Their difference $[4, 0, 0, 4, 0]$ is the trace vector for the space of 4 quadrics. This is an actual group character for H_8 . More precisely, $[4, 0, 0, 4, 0]$ is the sum of characters of the four one dimensional irreps $V_1..V_4$. This is why Beauville picked q_i from the direct sum of copies of V_i in $\text{Sym}^2(V)$.

The only other non abelian group of order 8 is the dihedral group D_8 . It's natural to ask that whether D_8 acts freely on any smooth complete intersections of 4 quadrics in \mathbb{P}^7 . Dihedral group D_8 is presented by $\{a, b | a^4 = 1, b^2 = 1; ab = ba^3\}$. It has 5 conjugacy classes $\{(1), (b), (ab), (a^2), (a)\}$. Again, we identify V with $H^0(X, \mathcal{O}(1))$, and we assume $\mathcal{O}(1)$ can be linearized so that D_8 acts on V . If D_8 acts freely on X , the trace vector of V should be $[8, 0, 0, 0, 0]$ i.e. V must be the regular representation. The trace vector for $\text{Sym}^2(V)$ is then $[36, 4, 4, 4, 0]$. Subtracting $[32, 0, 0, 0, 0]$, we get $[4, 4, 4, 4, 0]$. It is *not* a group character. So D_8 can not act linearly on any smooth complete intersection of 4 quadrics in \mathbb{P}^7 .

For groups of order larger than 8 $\mathcal{O}(1)$ can't be linearized. Otherwise, the Holomorphic Lefschetz formula shows that the character of the action on $V = H^0(X, \mathcal{O}(1))$ is a fractional multiple of the character of the regular representation, which leads to a contradiction. Hence instead of linear representations we should look for projective representations. In next section, we will give a brief introduction to projective representations of finite groups. We will see how Holomorphic Lefschetz formula puts restriction on these projective representations.

3. PRELIMINARIES OF PROJECTIVE REPRESENTATIONS

In the first part of this section we recall some facts about projective representations of finite groups. Our notations follow[Ber]. After that we define the allowable action of a subgroup of $\mathbb{PGL}(8, \mathbb{C})$.

Definition 3.1. Let G be a finite group. A triple (Γ, f, A) is called a central extension of G if Γ is a group, $A \subseteq Z(\Gamma)$ and f is a homomorphism of Γ onto G such that $\ker f = A$. A central extension (Γ, f, A) is called Schur Cover of G if A equals the second group homology $H_2(G, \mathbb{Z})$; this homology group is called Schur multiplier of G .

Theorem 3.2. [Ber] *If (Γ, f, A) is a Schur cover of G , then every projective representation P of G lifts to a linear representation of Γ . Conversely, any linear representation of Γ where A acts by scalar matrices is a lift of a projective representation of G .*

Remark 3.3. Schur multiplier is an invariant of G while the Schur cover is not uniquely defined. But by last theorem, given a Schur cover of G , all the projective representations of G can be realized by linear representations of Γ .

Definition 3.4. Two projective representations of G are called projective equivalent if they are conjugated in $\mathbb{PGL}(n, \mathbb{C})$.

Any projective representation of G is given by a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{GL}(8, \mathbb{C}) & \longrightarrow & \mathbb{PGL}(8, \mathbb{C}) & \longrightarrow & 1 \\ \uparrow & & \uparrow \tau & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where Γ is a Schur cover of G . Usually the map τ is not injective. Consider the short exact sequence:

$$0 \longrightarrow K/\text{Ker}(\tau) \longrightarrow \Gamma/\text{Ker}(\tau) \longrightarrow G \longrightarrow 1$$

Here $K/\text{Ker}(\tau)$ is a cyclic group. By theorem 3.2, projective representations of G are in one to one correspondence with linear representations of $\Gamma/\text{Ker}(\tau)$.

Definition 3.5. We say that a finite group $G \subset \mathbb{PGL}(8, \mathbb{C})$ has an allowable action if G acts freely on some smooth complete intersection X of four quadrics in \mathbb{P}^7 . We will call the correspondent G -action linear allowable action if G can be lifted to $\mathbb{GL}(8, \mathbb{C})$. Similarly, if G acts freely on some complete intersections X of four quadrics with at most ordinary double points, we say that G has a semi-allowable action.

Proposition 3.6. *If G has an allowable or semi-allowable action then $|G|$ divides 256.*

Proof. In [BK], Browder and Katz proved a general theorem about free action of finite groups on projective varieties:

Theorem 3.7. [BK] *Let X be a projective variety in \mathbb{P}^n and G is a finite subgroup of $\mathrm{PGL}(n+1, \mathbb{C})$. If G acts freely on X then, $|G|$ divides the square of the degree of X .*

We are considering complete intersections of four quadrics X in \mathbb{CP}^7 , which have degree 16. By theorem of Browder and Katz, if G acts freely on X then $|G|$ divides 256. \square

Remark 3.8. In section 5 we are going to argue the maximal order of G is 64.

If G has an allowable action on X , then it has an induced projective action on $H^0(X, \mathcal{O}(1))$. We denote this vector space by V . By theorem 3.2, the group $\Gamma/Ker(\tau)$ acts linearly on V with the cyclic subgroup $K/Ker(\tau)$ acting by scalar matrices. By Holomorphic Lefschetz formula, all the elements in Γ not in $K/Ker(\tau)$ have trace 0. If we fix a generator σ of $K/Ker(\tau)$ of order 2^d , then σ should act on V as a scalar matrix ξI where ξ is a primitive 2^d -th root of unity and I stands for identity matrix. Denote the trace vector of Γ for a given representation V by t_V^Γ . All the entries in t_V^Γ are 0 except ones corresponding to the conjugacy classes $\{(\sigma^k), k = 0, 1, \dots, 2^d - 1\}$. These conjugacy classes have trace $8\xi^k$. Similarly entries of $t_{H^0(X, \mathcal{O}(2))}^\Gamma$ are $32\xi^{2k}$ for conjugacy classes $\{(\sigma^k), k = 0, 1, \dots, 2^d - 1\}$ and 0 otherwise. From t_V^Γ we can compute the trace vector of the induced representation $Sym^2(V)$ and we denote it by $t_{Sym^2(V)}^\Gamma$. Vector $v = t_{Sym^2(V)}^\Gamma - t_{H^0(X, \mathcal{O}(2))}^\Gamma$ is the trace vector for the space of four quadrics. The assumption that G acts freely on X will force vectors t_V^Γ and v to be group characters.

Definition 3.9. We say a central extension

$$0 \longrightarrow K/Ker(\tau) \longrightarrow \Gamma/Ker(\tau) \longrightarrow G \longrightarrow 1$$

satisfies Lefschetz condition if the trace vectors t_V^Γ and v defined above are both group characters.

Proposition 3.10. *If G has a semi-allowable action then it satisfies Lefschetz condition.*

Proof. Use Holomorphic Lefschetz formula for $\pi^*(\mathcal{O}(1))$ and $\pi^*(\mathcal{O}(2))$ on the blowup $\pi : \widehat{X} \rightarrow X$ of the singular locus of X . \square

Remark 3.11. A priori, Lefschetz condition is only necessary but not sufficient for G to have allowable action. We still need to check the fixed loci of G in \mathbb{P}^7 don't intersect with X to verify the freeness. However, in our cases all the groups satisfying Lefschetz condition are actually allowable for $|G| < 64$. For $|G| = 64$ the necessity of Lefschetz condition follows from the fact that ordinary double points are rational singularities. Details are left to the readers.

4. CLASSIFICATION ALGORITHM

Our target is to classify the allowable and semi-allowable actions on complete intersections of four quadrics in \mathbb{P}^7 up to projective equivalence. In this section we describe the scheme of our algorithm. In Appendix we supply the GAP codes of the algorithm.

Recall that every projective representation gives a commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(8, \mathbb{C}) & \longrightarrow & \mathrm{PGL}(8, \mathbb{C}) \longrightarrow 1 \\
 \uparrow & & \uparrow \tau & & \uparrow & & \uparrow \\
 0 & \longrightarrow & K & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1
 \end{array}$$

where Γ is a Schur cover of G and K is its Schur Multiplier. Generally K is quite big but the exponent of K is controlled by order of G by the following lemma.

Lemma 4.1. *Let G be a finite group and K be its Schur Multiplier. Denote e for exponent of K . Then $e^2 \mid |G|$.*

Proof. [Ber] □

This lemma tells us the cyclic group $K/\mathrm{Ker}(\tau)$ in the central extension

$$0 \longrightarrow K/\mathrm{Ker}(\tau) \longrightarrow \Gamma/\mathrm{Ker}(\tau) \longrightarrow G \longrightarrow 1$$

has order at most 8. By theorem 3.2, given a group G of order no larger than 64, all projective representations of G are realized by such extensions.

Now we will describe the algorithm for $|G| = 64$. Lower order groups are handled similarly.

Lemma 4.2. *If $|G| = 64$ and G acts freely on X then $|K/\mathrm{Ker}(\tau)| \geq 4$.*

Proof. If $K/\mathrm{Ker}(\tau)$ has order 2 then the sheaf $\mathcal{O}(2)$ is G linearizable. This implies $\dim(H^0(X, \mathcal{O}(2)))$ is divisible by 64. But $H^0(X, \mathcal{O}(2))$ has dimension 32. □

Following this lemma, we need to consider projective representations of a 64 group G given by the following two types of central extensions.

- (1) A 256-group H with a subgroup $\mathbb{Z}/4$ acting as diagonal matrix $\xi^2 I$;
- (2) A 512-group H with a subgroup $\mathbb{Z}/8$ acting as diagonal matrix ξI .

Here I represents the 8×8 identity matrix and ξ is a primitive 8th root of unity.

Step I: Check the Lefschetz condition for the central extensions

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow H \longrightarrow G \longrightarrow 1$$

and

$$0 \longrightarrow \mathbb{Z}/8 \longrightarrow H \longrightarrow G \longrightarrow 1$$

Let H go over all groups of order 256 and 512 and output all G that satisfy Lefschetz condition. We use the GAP library of finite groups of small order. There are 56092 different groups of order 256 and 10494213 order 512 groups.

Step II: For each group G that appears in Step I, compute all possible extensions of G of the form:

$$0 \longrightarrow K/Ker(\tau) \longrightarrow \Gamma/Ker(\tau) \longrightarrow G \longrightarrow 1$$

for a fixed Schur Cover Γ . This can be done by computing kernels of all the group characters of K . By theorem 3.2, such extensions are 1-1 correspondent with nonequivalent projective representations.

Step III: check Lefschetz condition on extensions of Step II and output those extensions that satisfy it.

Step IV: check the fixed loci of the groups obtained from Step III and show they don't intersect X .

Step V: check that the generic complete intersection has at most ODP singularities in the semi-allowable case or is smooth in the allowable case. The final output is a list of projective representations of groups with allowable or semi-allowable actions. The same group might appear on this list for several times with different projective representations. Step I, II and III were done in GAP. Steps IV and V were done in Macaulay. The results of these calculations are presented in the next section.

5. RESULTS

In this section we present the results of the algorithm of the last section.

Remark 5.1. Many group theoretic computation in this paper are done in GAP. It has a small group library where all groups of given order less than 2000 are listed. For instance the quaternion group H_8

is represented by (8,4) in GAP library, where 8 for its order and 4 for its index in GAP library.

There are 8 nontrivial groups of order less and equal to 8. We will see all of them have allowable actions except the dihedral group D_8 . Further all the order 8 allowable action are linear.

There are 14(resp. 51) non-isomorphic 16-groups(resp. 32-groups). In the following tables we list all the allowable groups by their indices, together with the extension $\Gamma/Ker(\tau)$ representing the correspondent projective representation. We also give number of allowable actions up to projective equivalence.

When the order is less than 64, the generic element of the family with allowable action is a smooth complete intersection of four quadrics in \mathbb{P}^7 . However this is no longer true for 64-groups.

There are 267 different groups of order 64. Of these 267 groups there are five groups that are semi-allowable.

Remark 5.2. We want to say a little bit more about these five 64-groups. The group (64,2) is the abelian group $\mathbb{Z}/8 \times \mathbb{Z}/8$. Its Schur cover is the Heisenberg group $(\mathbb{Z}/8)^2 \ltimes \mathbb{Z}/8$. The group (64,3) is a semi-direct product of two copies of $\mathbb{Z}/8$ and (64,179) is a semi-direct product of quaternion group H_8 and $\mathbb{Z}/8$. These first 3 groups all contain a maximal abelian subgroup $\mathbb{Z}/4 \times \mathbb{Z}/8$, which has GAP index (32,3). It was observed in [BH] that these three 64-groups act on the same family. This is a subfamily of the 3 dimensional family with (32,3) action, which is invariant under additional involutions.

The other two groups (64,68) and (64,72) don't have obvious semi-direct product structures. Both of them contain a maximal abelian subgroup $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$, which has GAP index (32,21). These two groups act on another family (Theorem 6.3).

Remark 5.3. Observe that all groups listed in Table 1 are subgroups of these five 64-groups with only two exceptions: (32,4) and (32,5). Furthermore in (32,2) case, we are not sure that both projective representations are induced from representations of 64-groups. It turns out (from Macaulay) all the actions for $|G| \leq 32$ in Table 1 are allowable. For $|G| = 64$, they are semi-allowable.

Remark 5.4. The readers might observe that the 32-group (32,2) and the 64-group (64,72) have two different projective representations, i.e. there are two non-conjugated embeddings of these finite groups into $\text{PGL}(8, \mathbb{C})$. Recall that projective representations are one to one correspondent with central extensions of G . They are quotient groups of some Schur Cover. It is a natural question to ask that whether these

TABLE 1. (semi-)allowable action of order 2 to 64

Groups	Extension	Schur Multiplier
$\mathbb{Z}/2$	$\mathbb{Z}/2$	id group
$\mathbb{Z}/4$	$\mathbb{Z}/4$	id group
$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	id group
$\mathbb{Z}/8$	$\mathbb{Z}/8$	id group
$\mathbb{Z}/2 \times \mathbb{Z}/4$	$\mathbb{Z}/2 \times \mathbb{Z}/4, (16,3)$	$\mathbb{Z}/2$
$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
H_8	H_8	id group
(16,2)	(64,18)	$\mathbb{Z}/4$
(16,4)	(32,14)	$\mathbb{Z}/2$
(16,5)	(32,5)	$\mathbb{Z}/2$
(16,10)	(32,22)	$(\mathbb{Z}/2)^3$
(16,12)	(32,29)	$(\mathbb{Z}/2)^2$
(32,2)	(64,18), (64,23)	$(\mathbb{Z}/2)^3$
(32,3)	(128,6)	$\mathbb{Z}/4$
(32,4)	(64,28)	$\mathbb{Z}/2$
(32,5)	(64,4)	$(\mathbb{Z}/2)^2$
(32,13)	(64,46)	$\mathbb{Z}/2$
(32,21)	(128,462)	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/4$
(32,35)	(64,182)	$(\mathbb{Z}/2)^2$
(32,47)	(64,224)	$(\mathbb{Z}/2)^5$
(64,2)	$(\mathbb{Z}/8)^2 \rtimes \mathbb{Z}/8$	$\mathbb{Z}/8$
(64,3)	(256,321)	$\mathbb{Z}/4$
(64,68)	(256,4235)	$\mathbb{Z}/2 \times \mathbb{Z}/4$
(64,72)	(256,4222), (256,4233)	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/4$
(64,179)	(256,6447)	$\mathbb{Z}/4$

two representation can be identified by some outer automorphism of the group G . It turns out that the two different projective representations of (64,72) are identified by some outer automorphism of (64,72). In other words, these are two different ways of parametrizing the same subgroup of $\mathbb{PGL}(8, \mathbb{C})$ (see section 6).

By proposition 3.6 the maximal order of allowable action we can get is 256. Suppose there is an order 128 semi-allowable group. Then all its 64 subgroups must be semi-allowable. By a GAP calculation we check that there are no 128-groups, all of whose order 64 subgroups are among $\{(64, 2), (64, 3), (64, 68), (64, 72), (64, 179)\}$. Hence there is no (semi-)allowable group of order higher than 64.

6. COMPLETE INTERSECTION VARIETIES

In the last section we found five semi-allowable 64-groups. In the following two theorems we will show three of them act freely on a two dimensional family of complete intersections of 4 quadrics in \mathbb{P}^7 , and the other two groups act freely on a different dimension two family.

Theorem 6.1. *Let X be complete intersection of four quadrics:*

$$\begin{aligned} q_1 &= t_1(x_1^2 + x_5^2) + t_2(x_2x_8 + x_4x_6) + t_3x_2x_7 \\ q_2 &= t_1(x_2^2 + x_6^2) + t_2(x_3x_1 + x_5x_7) + t_3x_3x_8 \\ q_3 &= t_1(x_3^2 + x_7^2) + t_2(x_4x_2 + x_6x_8) + t_3x_4x_1 \\ q_4 &= t_1(x_4^2 + x_8^2) + t_2(x_5x_3 + x_7x_1) + t_3x_5x_2. \end{aligned}$$

We define the groups G_1, G_2, G_3 as subgroups of $\mathbb{PGL}(8, \mathbb{C})$. Group G_1 is generated by τ and the permutation $\sigma = (12345678)$ of the coordinates x_i , where $\tau(x_i) = \xi^{i-1}x_i$ with ξ a primitive 8th root of unity. Group G_2 generated by τ and $\sigma_1 = (18325476)$. Then G_2 is a nonabelian group isomorphic to a semidirect product of two copies of $\mathbb{Z}/8\mathbb{Z}$. Group G_3 is generated by τ and the permutations $\sigma_2 = (1357)(2468)$ and $\sigma_3 = (1256)(4387)$. Then G_3 is a nonabelian group isomorphic to a semidirect product of normal subgroup $\mathbb{Z}/8\mathbb{Z}$ generated by τ and the quaternion group H generated by σ_2 and σ_3 . Groups G_1, G_2, G_3 act freely on X . As in remark 6.3, $G_1 = (64, 2)$, $G_2 = (64, 3)$ and $G_3 = (64, 179)$.

Proof. See [BH]. □

Now we state the theorem for the groups $(64, 68)$ and $(64, 72)$. First define groups G_4, G_5, G_5' as subgroups of $\mathbb{GL}(8, \mathbb{C})$. Group G_4 generated by coordinates transformations $\sigma_1, \sigma_2, \sigma_3$, where
 $\sigma_1 : (x_1, \dots, x_8) \mapsto (\xi x_7, \xi x_8, \xi^3 x_5, \xi^3 x_6, -\xi x_3, -\xi x_4, \xi^3 x_1, \xi^3 x_2)$.
 $\sigma_2 : (x_1, \dots, x_8) \mapsto (-x_2, ix_1, -x_4, -ix_3, -ix_6, x_5, ix_8, x_7)$.
 $\sigma_3 : (x_1, \dots, x_8) \mapsto (\xi^3 x_5, -\xi^3 x_6, -\xi x_7, \xi x_8, \xi^3 x_1, -\xi^3 x_2, -\xi x_3, \xi x_4)$
Group G_5 is generated by $\sigma_3, \sigma_4, \sigma_5$ where
 $\sigma_4 : (x_1, \dots, x_8) \mapsto (\xi x_7, \xi x_8, -\xi^3 x_5, \xi^3 x_6, -\xi x_3, -\xi x_4, \xi^3 x_1, -\xi^3 x_2)$.
 $\sigma_5 : (x_1, \dots, x_8) \mapsto (\xi^3 x_6, \xi^3 x_5, -\xi x_8, \xi x_7, \xi^3 x_2, \xi^3 x_1, \xi x_4, -\xi x_3)$
Group G_5' is generated by $\sigma_3, \sigma_4, \xi \sigma_5$. These three groups G_4, G_5, G_5' are respectively $(256, 4235)$, $(256, 4222)$ and $(256, 4233)$. The corresponding projective groups in $\mathbb{PGL}(8, \mathbb{C})$ are $(64, 68)$ and $(64, 72)$. The last two groups G_5, G_5' have the same projective group $(64, 72)$. By abusing notations we denote G_4, G_5 also for the projective groups.

Remark 6.2. These two groups G_5, G_5' lead to two nonequivalent projective representations of $(64, 72)$ that were listed in Table 1. However,

the corresponding subgroups of $\mathbb{PGL}(8, \mathbb{C})$ are the same, and these two representations only differ by an outer automorphism of (64,72).

Theorem 6.3. *Let X be a complete intersection of four quadrics in \mathbb{P}^7 cut out by:*

$$\begin{aligned} q_1 &= t_1(x_1^2 + x_2^2) - t_2(x_3^2 + x_4^2) + t_1(x_5^2 + x_6^2) + t_2(x_7^2 + x_8^2) \\ q_2 &= -t_2(x_1^2 + x_2^2) + t_1(x_3^2 + x_4^2) + t_2(x_5^2 + x_6^2) + t_1(x_7^2 + x_8^2) \\ q_3 &= s_1(x_1^2 - x_2^2) - s_2(x_3^2 - x_4^2) + s_1(x_5^2 - x_6^2) + s_2(x_7^2 - x_8^2) \\ q_4 &= -s_2(x_1^2 - x_2^2) + s_1(x_3^2 - x_4^2) + s_2(x_5^2 - x_6^2) + s_1(x_7^2 - x_8^2). \end{aligned}$$

Groups G_4, G_5 act freely on X . Furthermore $G_4 = (64, 68), G_5 = (64, 72)$.

Proof. We will prove the theorem for G_4 . The argument for G_5 is similar. Consider central extension

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow (256, 4235) \longrightarrow G_4 \longrightarrow 1$$

Group (256,4235) has 46 irreducible representations, indexed by $X_1 \dots X_{46}$, where X_1, \dots, X_{16} are 1-irreps, X_{17}, \dots, X_{44} are 2-irreps and X_{45}, X_{46} are 8-irreps. Again we first identify V with $H^0(X, \mathcal{O}(1))$. By Holomorphic Lefschetz formula, we must pick V to be the irreducible representation X_{45} . The second symmetric product V has decomposition:

$$\text{Sym}^2(V) = \bigoplus_{i \in I} X_i \oplus X_{35}^{\oplus 2} \oplus X_{36}^{\oplus 2}$$

$$I = \{19, 20, 21, 22, 25, 26, 27, 28, 33, 34, 41, 42, 43, 44\}.$$

The four dimensional space of quadrics has decomposition $X_{35} \oplus X_{36}$. Pick a basis (x_1, \dots, x_8) for $V = X_{45}$. We get an induced basis for $\text{Sym}^2(V)$. They are represented by homogenous quadratic equations in x_1, \dots, x_8 . In particular, $X_{35}^{\oplus 2} = \langle x_1^2 + x_2^2 + x_5^2 + x_6^2, x_3^2 + x_4^2 + x_7^2 + x_8^2 \rangle \oplus \langle x_7^2 + x_8^2 - x_3^2 - x_4^2, x_5^2 + x_6^2 - x_1^2 - x_2^2 \rangle$. Respectively, $X_{36}^{\oplus 2} = \langle x_1^2 - x_2^2 + x_5^2 - x_6^2, x_3^2 - x_4^2 + x_7^2 - x_8^2 \rangle \oplus \langle x_7^2 - x_8^2 - x_3^2 + x_4^2, x_5^2 - x_6^2 - x_1^2 + x_2^2 \rangle$. These give the cut out equations (7.1). From these equations we can view the parameter space of this two dimensional family of X as a subset in $\mathbb{P}^1 \times \mathbb{P}^1$ where $(t_1 : t_2)$ and $(s_1 : s_2)$ are homogeneous coordinates of each \mathbb{P}^1 .

To show G_4 acts without fixed points, we need to check the intersection of the fix loci of all conjugacy classes of G_4 with X . The group G_4 acts freely if these intersections are empty for generic choice of coefficients t_1, t_2, s_1, s_2 . This can be done very easily in Macaulay 2. \square

Remark 6.4. In section 5 we mentioned (64,72) has two different projective representations (256, 4222), (256, 4233). A calculation shows both of them act freely on this family.

From now on, we will focus on the two families in theorem 6.1 and 6.3. A simple Macaulay calculation show a generic element X has 64 ordinary double points, i.e. the 64-group actions are semi-allowable.

7. RESOLUTIONS OF SINGULARITIES

Let's first review the main theorem in [BH]. Let X be a complete intersection of four quadrics cut out by equations in theorem 6.1. We have seen in last section three 64-groups $(64,2)$, $(64,3)$ and $(64,179)$ act freely on X . This variety was first discovered by Gross and Popescu. In [GPO], they studied the birational geometry of X , including the resolution of singularities. They have proved the following theorem for the group $(64,2)$.

Theorem 7.1. *The singular Calabi-Yau 3-fold X has an equivariant small projective resolution \tilde{X} , i.e. \tilde{X} is a smooth projective Calabi-Yau 3-fold with free actions by $(64,2)$, $(64,3)$ and $(64,179)$. The resolution \tilde{X} has Hodge numbers $h^{1,1} = 2, h^{1,2} = 2$. Furthermore, \tilde{X} contains a pencil of abelian surfaces with polarization $(1,8)$.*

In this section we obtain a similar result for the family defined in theorem 6.3. We will prove the generic element X in this family also has an equivariant small projective resolution. Recall X is cut out by equations:

$$\begin{aligned} q_1 &= t_1(x_1^2 + x_2^2) - t_2(x_3^2 + x_4^2) + t_1(x_5^2 + x_6^2) + t_2(x_7^2 + x_8^2) \\ q_2 &= -t_2(x_1^2 + x_2^2) + t_1(x_3^2 + x_4^2) + t_2(x_5^2 + x_6^2) + t_1(x_7^2 + x_8^2) \\ q_3 &= s_1(x_1^2 - x_2^2) - s_2(x_3^2 - x_4^2) + s_1(x_5^2 - x_6^2) + s_2(x_7^2 - x_8^2) \\ q_4 &= -s_2(x_1^2 - x_2^2) + s_1(x_3^2 - x_4^2) + s_2(x_5^2 - x_6^2) + s_1(x_7^2 - x_8^2). \end{aligned}$$

The jacobian matrix of X is

$$\begin{pmatrix} t_1x_1 & t_1x_2 & -t_2x_3 & -t_2x_4 & t_1x_5 & t_1x_6 & t_2x_7 & t_2x_8 \\ -t_2x_1 & -t_2x_2 & t_1x_3 & t_1x_4 & t_2x_5 & t_2x_6 & t_1x_7 & t_1x_8 \\ s_1x_1 & -s_1x_2 & -s_2x_3 & s_2x_4 & s_1x_5 & -s_1x_6 & s_2x_7 & -s_2x_8 \\ -s_2x_1 & s_2x_2 & s_1x_3 & -s_1x_4 & s_2x_5 & -s_2x_6 & s_1x_7 & -s_1x_8 \end{pmatrix}$$

A point on X is singular if and only if the jacobian matrix is degenerated at this point. First let us compute the singular locus of X .

Lemma 7.2. *A point $P \in X$ is singular if and only if exactly four coordinates out of (x_1, \dots, x_8) are zero.*

Proof. Let $P = (x_1 : \dots : x_8)$ be a point on X . Observe that P has at most 4 zeros in coordinates because otherwise P can't sit on X with generic t_i and s_i . We first prove if P has 4 zeros then it must be a singular point.

Let μ be a combination of 4 distinct numbers from 1 to 8, call its complement by $\bar{\mu}$. Denote P_μ be a point with $\{x_i = 0 | i \in \mu\}$ and $J_{\bar{\mu}}$ be the four by four minor of the jacobian matrix by picking the columns from $\bar{\mu}$. Since all equations $q_1 \dots q_4$ consist of square terms, the

jacobian matrix J is equivalent with the coefficient matrix for $q_1..q_4$ up to elementary transformation. So $J_{\bar{\mu}}$ degenerates if and only if $q_1..q_4$ has nonzero solution by plugging in $\{x_i|i \in \mu\}$. This proves the first direction.

If P is a singular point, we pick a 4-combination μ such that $\{x_i \neq 0|i \in \mu\}$ for P . Since P is singular the jacobian matrix J evaluated at P degenerates. In particular, J_{μ} degenerates. If the rest coordinates $\{x_i|i \in \bar{\mu}\}$ don't vanish simultaneously we will get one parameter family of solutions along which the jacobian matrix degenerates. However a simple Macaulay computation shows the singular loci has dimension 0. So all the rest coordinates must be zero. \square

For X , (1468), (1367), (1457), (2467), (2357), (2458), (1358), (2368) are 8 combinations corresponding to singular points P_{μ} . Since X is cut out by degree two equations, if we let four coordinates to be zero the rest four variables satisfy four quadratic equations. There are 8 solutions up to scaling. Hence each combinations give 8 singular points. These 64 points form a group orbit, for both (64,68) and (64,72). We will see these 64 singularities are ordinary double points. Let's fix a combination, say (1468). The singular points that corresponding to this combination has form $(0 : y_2 : y_3 : 0 : y_5 : 0 : y_7 : 0)$. Solve q_1, \dots, q_4 by plugging in $y_1 = y_6 = y_4 = y_8 = 0$ we get:

$$\begin{aligned} q_1 &= t_1 y_2^2 - t_2 y_3^2 + t_1 y_5^2 + t_2 y_7^2 = 0 \\ q_2 &= -t_2 y_2^2 + t_1 y_3^2 + t_2 y_5^2 + t_1 y_7^2 = 0 \\ q_3 &= -s_1 y_2^2 - s_2 y_3^2 + s_1 y_5^2 + s_2 y_7^2 = 0 \\ q_4 &= s_2 y_2^2 + s_1 y_3^2 + s_2 y_5^2 + s_1 y_7^2 = 0. \end{aligned}$$

After we solve t_1, t_2, s_1, s_2 by y_i , we can rewrite the original equations q_1, \dots, q_4 as:

$$\begin{aligned} q_1 &= (y_3^2 - y_7^2)(x_1^2 + x_2^2) - (y_2^2 + y_5^2)(x_3^2 + x_4^2) \\ &\quad + (y_3^2 - y_7^2)(x_5^2 + x_6^2) + (y_2^2 + y_5^2)(x_7^2 + x_8^2) \\ q_2 &= -(y_2^2 + y_5^2)(x_1^2 + x_2^2) + (y_3^2 - y_7^2)(x_3^2 + x_4^2) \\ &\quad + (y_2^2 + y_5^2)(x_5^2 + x_6^2) + (y_3^2 - y_7^2)(x_7^2 + x_8^2) \\ q_3 &= (y_3^2 - y_7^2)(x_1^2 - x_2^2) - (y_5^2 - y_2^2)(x_3^2 - x_4^2) \\ &\quad + (y_3^2 - y_7^2)(x_5^2 - x_6^2) + (y_5^2 - y_2^2)(x_7^2 - x_8^2) \\ q_4 &= -(y_5^2 - y_2^2)(x_1^2 - x_2^2) + (y_3^2 - y_7^2)(x_3^2 - x_4^2) \\ &\quad + (y_5^2 - y_2^2)(x_5^2 - x_6^2) + (y_3^2 - y_7^2)(x_7^2 - x_8^2). \end{aligned}$$

Additionally y_2, y_3, y_5, y_7 satisfy relation $y_3^4 - y_7^4 = y_2^4 + y_5^4$.

These computations give the relations between the space of singular points and the family of complete intersection of quadrics. The

positions of the 64 singular points uniquely determine the family of complete intersections.

Let's denote G for $(64, 68)$.

Theorem 7.3. *There exist a G -equivariant small resolution \tilde{X} by blowing up a smooth G -invariant abelian surface in X .*

Proof. To get such small resolution, we need to find a Weil but not Cartier divisor, which is invariant under the group action. By blowing up this divisor we get the projective small resolution. Consider the subvariety cut out by the following two equations.

$$\begin{aligned} f_1 &= r_1x_1x_2 - r_2x_3x_4 + r_1x_5x_6 + r_2x_7x_8 \\ f_2 &= -r_2x_1x_2 + r_1x_3x_4 + r_2x_5x_6 + r_1x_7x_8 \end{aligned}$$

Notice equations $\{x_1x_2 + x_5x_6, x_3x_4 + x_7x_8\}$ span the 2-irreps X_{33} and $\{x_1x_2 - x_5x_6, x_3x_4 - x_7x_8\}$ span 2-irreps X_{34} . So f_1, f_2 are two generic elements in $X_{33} \oplus X_{34}$. Equations f_1, f_2 together with q_1, \dots, q_4 cut out an G invariant smooth surface in X , denoted by S_{r_1, r_2} . By Macaulay calculation, under generic choices of r_1 and r_2 , this is a smooth abelian surface. Notice this abelian surface cannot be cut out by one equation. Actually if we pick one of two equations f_1 and f_2 we will get unions of two abelian surfaces. Hence S_{r_1, r_2} is a Weil divisor but not Cartier. The abelian surface S_{r_1, r_2} has Hilbert polynomials $P(z) = 8z^2$. This tells us it has arithmetic genus $p_a = -1$, i.e. it is of degree 16 in \mathbb{P}^7 . Any two such surfaces S_{r_1, r_2} intersect at the 64 singular points. And from these cut out equations we see the singularities must be ordinary double points. By blowing up S_{r_1, r_2} , we get a smooth Calabi-Yau variety \tilde{X} . Because S_{r_1, r_2} is G -invariant \tilde{X} also has a free G -action. \square

Remark 7.4. For the group $(64, 72)$, we need to blowup a different Weil divisor S_{r_1, r_2} cut out by equations:

$$\begin{aligned} f_1 &= r_1x_1x_5 - r_2x_2x_6 + r_1x_3x_7 - r_2x_4x_8 \\ f_2 &= -r_1x_1x_5 + r_2x_2x_6 + r_1x_3x_7 - r_2x_4x_8 \end{aligned}$$

Recall that there are two different allowable action of $(64, 72)$, lifted to $G_5 = (256, 4222)$ and $G_5' = (256, 4233)$. Both of them act on S_{r_1, r_2} , i.e. these two actions have the same equivariant resolutions.

To make it precise, we denote the resolution of X corresponding to G -allowable action by \tilde{X}_G .

Corollary 7.5. *The quotient variety \tilde{X}_G/G is a smooth Calabi-Yau 3-fold with fundamental group G . Here $G = (64, 68)$ or $(64, 72)$.*

Finally we study the fibration structure of this family. As we see in [BH],[GPo] and [GPa] these Calabi-Yau 3-folds are closely related to moduli space of polarized abelian surfaces.

Proposition 7.6. *The equations f_1, f_2 form a sub linear system of dimension 1 of $\mathcal{O}(2)$ with 64 base points exactly at the 64 ordinary double points.*

Proof. We need to show $\phi : x \mapsto (f_1(x) : f_2(x))$ is a rational map defined except the 64 ODP's. It is obvious ϕ is defined at $X \setminus S_{r_1, r_2}$. For any point on S_{r_1, r_2} which is not the ordinary double point, f_1 and f_2 have a common divisor, i.e. S_{r_1, r_2} is cut out locally by one equation. So ϕ can be extended to X except the 64 ordinary double points. \square

Remark 7.7. Consider the space of quadrics spanned by q_1, \dots, q_4 together with f_1, f_2 . These equations cut out a (2,4) polarized abelian surfaces in \mathbb{P}^7 ([Ba]). Any four linear independent equations cut out a Calabi-Yau complete intersection with 64 ordinary double points. However only a two dimensional subfamily has free G -action.

Remark 7.8. Let X be the Calabi-Yau 3-fold cut out by equations in theorem 7.2. Groups (64,68) and (64,72) act freely on X . It contains a pencil of (2,4) polarized abelian surfaces [Ba]. Say \tilde{X} is the small resolution by blowing up an smooth abelian surface introduced in theorem 7.3. This Calabi-Yau 3-fold \tilde{X} has Hodge number $h^{1,1} = 10, h^{1,2} = 10$. As we stated in the last remark, only a two dimensional subfamily in this ten dimensional family have G action. A similar argument to Remark 4.11 in [GPo] can be applied to compute the Hodge number of the quotient variety \tilde{X}/G . We expect that \tilde{X}/G has Hodge number $h^{1,1} = 2, h^{1,2} = 2$.

8. APPENDIX

Here we list the GAP codes for finding different projective representations and checking Lefschetz condition.

8.1. find projective representation.

```
G:=SmallGroup(-,-);
T:=EpimorphismSchurCover(G);
S:=Source(T);
K:=Kernel(T);
C:=CharacterTable(K);
N:=Order(K);
for i in [2..N] do
Ker:=KernelOfCharacter(Irr(C)[i]);
```

```

Q:=S/Ker;
Display(IdSmallGroup(Q));
od;
8.2. check the Lefschetz condition.
N:=16; M:=NumberSmallGroups(N); j:=0; for l in [1..M] do
G:=SmallGroup(N,l); C:=CharacterTable(G); cc:=ConjugacyClasses(C);
c:=Size(Irr(C)); for k in [2..c] do
v8:=0*[1..c];
v8[1]:=8;
v8[k]:=-8;
if IsCharacter(C,v8) then
    s2:=SymmetricParts(C,[v8],2)[1];
s2[1]:=4;
s2[k]:=4;
g:=Representative(cc[k]);
H:=Subgroup(G,[g]);
Q:=FactorGroup(G,H);
if IsCharacter(C,s2) then
    if j≠ 1 then
        Print;
        Display(IdSmallGroup(Q));
        j:=1;
    fi;
fi;
fi;
od; od;

```

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